Heat trace asymptotics on compact surface with conic singularity. Example of the tetrahedron.

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Contents

1 Introduction

The purpose of these notes is to show that it is reasonable to hope that one can detect the singular character of a manifold with conic singularities from the heat trace of some specific self-adjoint extension of the Hodge Laplacian acting on smooth differential forms supported away from the singularities. We compute the coefficients in the asymptotic expansion of the heat trace $tre^{-t\Delta_0}$ as $t \to 0+$ where Δ_0 is Laplace-Beltrami operator on a tetrahedron. Tetrahedron is an example of a compact Riemannian manifold with singularities. We see how the singularities contribute to the heat trace expansion.

The notes are organised as follows. In the introduction one can see the geometric description of the tetrahedron and the main result (1.1). In the Section 2 we consider un unbounded operator on a Hilbert space and discuss its self-adjoint extensions, in particular the construction of the Friedrichs extension is given. Also we define the Schatten class of operators in particular trace class. In the Section 3 we consider a two-dimensional compact manifold with one conic singularity and describe the Laplace-Beltrami operator on the regular part of it. For our purposes it is sufficient to consider only one singularity as it will be explained below. In the Section 4 we construct the heat trace expansion of Laplace-Beltrami operator on a compact subset of a Riemannian manifold. Finally, in the Section 5 we apply the theory developed in the Sections 2, 3 to the tetrahedron, combine in with the results from the Section 4 and derive the final result.

The geometry of the tetrahedron is the following. It is a compact oriented Riemannian flat manifold (meaning that the total curvature tensor is zero) with four conic singularities. The metric near the singularities is specified below. We assume that the length of every edge is equal to 1. Denote the singular points by A, B, C, D.

To deal with singular points we unfold the tetrahedron so that it forms a triangle on a plane. To do that we cut along AD, BD, CD and obtain the triangle such that the vertices correspond to D on the tetrahedron and the middle points of the edges correspond to A, B and C. Now we glue it together again in such a way that each point A, B, C, D is a tip of a cone over a circle with the length of the generatrix $\varepsilon > 0$. Than the radius of the circle, which is the base of the cone is $\varepsilon/2$, hence the inner angle of the cone is $\arcsin(1/2) = \frac{\pi}{6}$. Consider metric induced from $\mathbb{R}^3$. To obtain a coordinate system on the cone we take spherical coordinates with $r \in [0, \varepsilon)$ and $\theta \in [0, 2\pi)$

$$\begin{cases} x = r \sin \alpha \cos \theta \\ y = r \sin \alpha \sin \theta \\ z = r \cos \alpha \end{cases}$$

and let polar angle be constant. The metric is

$$g_{cone} = dr^2 + r^2 \sin^2 \alpha d\theta^2 = dr^2 + \frac{1}{4} r^2 d\theta^2.$$

Now the tetrahedron has the following structure. It is a compact Riemannian manifold $\overline{M} = \cup_{i=1}^4 \overline{U_i} \cup M_1$ with four conic singularities, where $\overline{U_i} := [0, \varepsilon) \times S^1$ with $\varepsilon > 0$ for $1, 2, 3, 4$ and M_1 is compact smooth manifold with boundary N which is a union of four circles S^1. The line in the notation is used to indicate that the manifold $\overline{M}$ is compact for the open manifold with the singular points excluded we omit the line and write M also $U_i := (0, \varepsilon) \times S^1$.

We develop all necessary details about the Laplace-Beltrami operator Δ_0 on the tetrahedron in the Sections 2 and 3. Here is the sketch of the result. For our purpose we deal with the operator Δ_0 separately on a neighbourhood of each singular point of the manifold $\Delta_0|_{U_i}$ and on the regular part $\Delta_0|_{M_1}$. For the operator on the regular part methods applicable for a smooth compact manifold are used. As for the singular parts we show that Laplace-Beltrami operator is unitary equivalent to a regular singular operator and use methods from [BS].

We choose a partition of unity on the tetrahedron φ_i, ψ with $i = 1, 2, 3, 4$ such that each $\varphi_i(r)$ has its support near one of the singularities and these functions depend only on r coordinate and ψ has its support outside the singularities. In these notes it is shown that for $1 \le i \le 4$

$$\mathrm{tr}\varphi_i e^{-t\Delta_0} \sim_{t\to 0+} \frac{1}{4} t^{-1} \int_0^\infty r\varphi_i(r) dr + \frac{1}{8} t^0$$

and

$$\mathrm{tr}\psi e^{-t\Delta_0} \sim_{t\to 0+} \frac{1}{4\pi} t^{-1} \int_M \psi vol_M$$

Hence on the tetrahedron there is an asymptotic

$$\mathrm{tr} e^{-t\Delta_0} \sim_{t\to 0+} \frac{\sqrt{3}}{4\pi} t^{-1} + \frac{1}{2} t^0. \tag{1.1}$$

The constant term in (1.1) reveals the presence of the singularities in the tetrahedron. For a moment assume we have a smooth flat two-dimensional compact Riemannian manifold M than as we see in the Section 4 the heat trace expansion would have the form $tre^{-t\Delta_0} \to_{t\to 0+} \frac{\mathrm{vol}(M)}{4\pi} t^{-1}$. In other words a smooth flat compact manifold does not have a constant term in its heat kernel expansion. Therefore the tetrahedron cannot be a smooth manifold.

2 Operators on a Hilbert space

2.1 Friedrichs' extension

Let H be a Hilbert space and A be a nonnegative densely defined symmetric operator on H. Now we construct the largest self-adjoint extension of A. The original work of Friedrichs is [F], the proof of the theorem can also be found in [RN] (Chap. VIII). To understand the construction we give the proof below following [Ga]. First, we develop some tools which we will need further.

Define two unitary operators $U, V : H \oplus H \to H \oplus H$ by $U(x, y) = (y, x)$ and $V(x, y) = -(y, x)$, then for any invertible operator A the operator U maps the graph of A to the graph of its inverse

$$U(\Gamma(A)) = \Gamma(A^{-1}). \tag{2.1}$$

Since U is unitary then for a subspace X

$$(UX)^{\perp} = U(X^{\perp}). \tag{2.2}$$

Proposition 2.1. *Let A be a densely defined operator on a Hilbert space H, then the following is true*

$$\Gamma(A^*) = V(\Gamma(A))^{\perp} = V(\Gamma(A)^{\perp})$$

Proof. The second inequality follows from (2.2).

First we proof $\Gamma(A^*) \subseteq V(\Gamma(A))^{\perp}$. Let $x \in D(A)$ and $y \in D(A^*)$

$$\langle V(x, Ax), (y, A^*y)\rangle = \langle -(Ax, x), (y, A^*y)\rangle = \langle -Ax, y\rangle + \langle x, A^*\rangle = 0.$$

Next we proof $V(\Gamma(A))^{\perp} \subseteq \Gamma(A^*)$. Let $x \in D(A)$ and $(y, z) \in \Gamma(A^*)$

$$\langle V(x, Ax), (y, z)\rangle = \langle -Ax, y\rangle + \langle x, z\rangle = 0.$$

We obtain $\langle Ax, y\rangle = \langle x, z\rangle$, hence $y \in D(A^*)$, $z = A^*y$ and $(y, z) \in \Gamma(A^*)$. □

Theorem 2.2. *(Riesz' theorem) Let $F : H \to \mathbb{C}$ be a continuous linear functional (i.e. $|F(x)| \leq c_F||x||$ for some $c_F > 0$ for any $x \in H$) then there is a unique $z_F \in H$ such that for any $x \in H$ we have $F(x) = \langle x, z_F\rangle$.*

Theorem 2.3. *(Friedrichs) A nonnegative densely defined symmetric operator A on a Hilbert space H has a nonnegative self-adjoint extension.*

Proof. Define a inner product on the domain $D(A)$

$$\langle x, y \rangle_A = \langle x, y \rangle + \langle Ax, y \rangle \tag{2.3}$$

for $x, y \in D(A)$. Recall that the corresponding norm is called *graph norm.* Let $\overline{D(A)}$ be a closure of the domain with respect to this norm. Since the operator is nonegative we have an inequality $\langle x, x \rangle \leq \langle x, x \rangle_A$, consequently $\overline{D(A)}$ is a completion with respect to the inner product $\langle \cdot, \cdot \rangle_A$.

Let $h \in H$ and $x \in \overline{D}(A)$. Consider a linear functional $F_h : x \mapsto \langle x, h \rangle$. It is bounded $||F_h x|| \leq ||x|| \cdot ||h|| \leq ||x||_A \cdot ||h||$, hense by Riesz theorem (Theorem 2.2) there is unique $Bh \in \overline{D(A)}$ such that for any $x \in \overline{D(A)}$ we have $F_h(x) = \langle x, Bh \rangle_A$. A map $B : H \to \overline{D(A)}$ is linear. Next we show that B is symmetric, nonnegative, injective and has a dense range, in particular self-adjoint and it has an inverse operator.

Symmetry

$$\langle Bx, y \rangle = F_y Bx = \langle Bx, By \rangle_A = \overline{\langle Bx, By \rangle_A} = \overline{F_x By} = \overline{\langle By, x \rangle} = \langle x, By \rangle$$

for any $x, y \in H$.

Positivity

$$\langle Bx, x \rangle = F_x Bx = \langle Bx, Bx \rangle_A \geq \langle Bx, Bx \rangle \geq 0$$

for $x \in H$.

Density of $R(B)$. Let $y \in \overline{D(A)}$ such that $\langle Bh, x \rangle$ for any $h \in H$, then

$$0 = \langle x, Bh \rangle = F_h x = \langle h, Bx \rangle,$$

hense $x = 0$.

Injectivity. Let $x \in H$ be such that $Bx = 0$, then $\forall y \in \overline{D(A)}$ we have

$$0 = \langle y, 0 \rangle_A = \langle y, Bx \rangle_A = F_x y = \langle x, y \rangle.$$

Since $\overline{D(A)}$ is dense, $x = 0$.

Denote the inverse of B by A_f. We will proof that $A_F := A_f - I_H$ is a self-adjoint extension of A. Note that A is surjective $A_f : D(A_f) \to H$ and $D(A_f \subset \overline{D(A)})$. Using Proposition 2.1 and an equality (2.1) for $A_f = B^{-1}$ we obtain

$$\Gamma(A_f^*) = (V(\Gamma(A_f)))^\perp = (V \circ U(\Gamma(B)))^\perp.$$

By the properties $V \circ U = -U \circ V$ and (2.2)

$$(V \circ U(\Gamma(B)))^\perp = -(U \circ V(\Gamma(B)))^\perp = -U((V(\Gamma(B)))^\perp).$$

Again applying Proposition 2.1 and using that B is self-adjiont

$$-U((V(\Gamma(B)))^{\perp}) = -U(\Gamma(B^*)) = -U(\Gamma(B)) = \Gamma(A_f).$$

Eventually we get $\Gamma(A_f^*) = \Gamma(A_f)$, i.e. A_f is self-adjoint.

Next we proof that A_f is nonnegative. Let $x \in D(A_f)$ and $x = By$

$$\langle x, Ax\rangle = \langle By, y\rangle = F_y By = \langle By, By\rangle_A \geq \langle By, By\rangle = \langle x, x\rangle \geq 0.$$

The domain of A_f is dense in $\overline{D(A)}$. Let $x \in \overline{D(A)}$ be orthogonal to $D(A_f)$ with respect to $\langle\cdot,\cdot\rangle_A$ and let $x' \in D(A_f)$, $x' = By'$

$$0 = \langle x, x'\rangle_A = \langle x, By'\rangle_A = F_{y'}x = \langle x, y'\rangle = \langle x, A_f x'\rangle$$

hense A_f is surjective $x = 0$.

Now we proof that A_F is an extension of A. Let $x, y \in D(A)$ then

$$\langle x, B(A + I_H)y\rangle_A = F_{(A+I_H)y}x = \langle x, (A + I_H)y\rangle = \langle x, (A + I_H)y\rangle = \langle x, y\rangle_A\,,$$

hense $\langle x, y - (A + I_H)y\rangle = 0$. Since $D(A)$ is dense in $D(B) = \overline{D(A)}$, for $y \in D(A)$ we have $B(A + I_H)y = y$, it means $y \in R(B) = D(A_f)$. Consequently $A_f y = A_f(B(A + I_H)y) = (A + I_H)y$. Eventually we obtain

$$A_F y = Ay \qquad \text{for any } y \in D(A).$$

□

2.2 Schatten classes and trace class

Let H be a Hilbert space and A compact operator on H then A^*A is compact and self-adjoint operator. Let

$$0 \leq s_1 \leq s_2 \leq \ldots$$

be eigenvalues of $(A^*A)^{1/2}$.

Definition 1. Let $1 \leq q < \infty$. A bounded operator A is *Schatten class operator of order* q and write $A \in S_q(H)$ if

$$||A||_q := \left(\sum_{i=1}^{\infty} s_i^q\right)^{1/q} < \infty.$$

Note that if $A \in S_q(H)$ then $||A||_q = tr(|A|^q) = tr((A^*A)^{1/2})^q$. If $A \in S_1(H)$ the operator A is *trace class* operator.

3 Geometrical operators on a non compact surface

In this section we consider a closed surface with one conic singularity. We remove the singular point and deal with a non compact manifold. The analysis on operators on a non compact manifold is complicated because operators are unbounded and may have many closed extensions. Below we define geometric operators on the surface and determine a Hilbert space where they are acting.

3.1 Operators $d, d^\dagger$ and Δ

Consider oriented 2-dimensional compact Riemannian manifold $\overline{M} = \overline{U} \cup M_1$ with one conic singularity where $\overline{U} = [0, \varepsilon) \times N$ with $\varepsilon > 0$ and M_1 is compact smooth manifold with boundary $N = S^1$. The line in the notation is used to indicate that the manifold $\overline{M}$ is compact for the open manifold with the singular point removed we omit the line and write M also $U = (0, \varepsilon) \times N$.

Consider space of smooth functions with compact support $C_c^\infty(M)$ on non-complete manifold M and space of differential one-forms with compact support $\lambda_c^1(M) := C_c^\infty(\Lambda T^* M)$. There are first order differential operators defined between the spaces exterior derivative $d : C_c^\infty \to \lambda_c^1(M)$ and its formal adjoint determined by the metric $d^\dagger : \lambda_c^1(M) \to C_c^\infty(M)$ furthermore we define Laplace-Beltrami operator on smooth functions with compact support $\Delta := d^\dagger d$. Sometimes we write Δ_0 to indicate that this operator acts on the space of functions.The operators d, $d^\dagger$ are densely defined in $L^2(M)$ and $L^2(\Lambda T^* M)$ correspondingly.

Proposition 3.1. *Operator $\Delta : L^2(M) \to L^2(M)$ is densely defined, symmetric and nonnegative.*

Proof. The operator is densely defined since both operators d and $d^\dagger$ are densely defined. Let $f \in C_c^\infty(M)$ then

$$(\Delta f, f) = (d^\dagger df, f) = (df, df) \geq 0$$

from which follow that Δ is symmetric and nonnegative. □

From the Proposition 3.1 and Theorem 2.3 follows that the operator Δ admits Friedrichs extension. In fact there can be many self-adjoint extensions of Δ.

3.2 Operators d_{min} and d_{max}

The operator d is a densely defined operator in the Hilbert space $L^2(M)$ acting on $C_c^\infty(M)$. Since the operator d has a densely defined adjoint $d^\dagger$ there exists at least one closed extension of d. Denote the closure of d by $d_{min} := \overline{d}$, i.e. the domain

$$\begin{aligned} D(d_{min}) := \{u \in L^2(M) \mid \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(M) : u_n \xrightarrow[n\to\infty]{} u, \\ du_n \xrightarrow[n\to\infty]{} v \in L^2(M)\} \end{aligned} \tag{3.1}$$

and $d_{min}u := v$.

Define $d_{max} := \overline{d^\dagger}^*$.

In principal if D is a closed extension of the operator d then

$$d_{min} \leq D \leq d_{max}$$

but we see in the next proposition that in case of the tetrahedron we have a unique closed extension of d.

The following general result on a Riemannian manifold of some dimension m connects the Friedrichs extension of Laplace-Beltrami operator to the extensions of d d_{min} and d_{max}.

Lemma 3.2. *Let E, F be hermitian vector bundles over a Riemannian manifold M and $d : C_c^\infty(E) \to C_c^\infty(F)$ be a differential operator. Denote by $d^\dagger$ its formal adjoint and $d^\dagger d$ a non negative symmetric operator on $L^2(E)$ with domain $C_c^\infty(M)$. Let $(d^\dagger d)^F$ be the Friedrichs extension of $d^\dagger d$. Then*

$$(d^\dagger d)^F = d^\dagger_{max} d_{min}$$

Proof. Both operators are self-adjoint hence it is sufficient to prove that

$$D((d^\dagger d)^F) \subset D(d^\dagger_{max} d_{min}).$$

Let $u \in D((d^\dagger d)^F)$ then $u \in D(d_{min})$. Since the Friedrichs extension is the maximal self-adjoint extension $u \in (d^\dagger d)_{max}$ consequently $d_{min}u \in D(d^\dagger_{max})$. □

Now let $M = U \cup M_1$ be a two-dimensional Riemannian manifold with conic singularity and let the metric on U be $g_U := dr^2 + r^2a^2d\theta^2, a > 0$. Note that we have $a = 1/2$ for the tetrahedron.

Proposition 3.3. *On a manifold with the metric near the singularity defined above we have a unique extension of d, i.e.*

1) $d_{min} = d_{max} := \tilde{d}$*;*

2) the operator $\tilde{\Delta} := \tilde{d}^{\dagger}\tilde{d}$ *is equal to the Friedrichs extension of* Δ*.*

Proof. 1) We need to proof that $D(d_{max}) \subset D(d_{min})$. Let $u \in D(d_{max})$ and φ be a function on M such that $\varphi \in C_c^{\infty}((-\varepsilon, \varepsilon))$ with $\varphi = 1$ near zero. Since $(1-\varphi) \in C_c^{\infty}(M)$ we have

$$(1-\varphi)u \in D(d_{max}).$$

Let $\varepsilon_1 \in (0, \varepsilon)$ and $U_{\varepsilon_1} := (0, \varepsilon_1) \times N$ then $M \setminus U_{\varepsilon_1}$ is complete and for sufficiently small ε_1 the restriction $(1-\varphi)|_{M \setminus U_{\varepsilon_1}}$ is zero near the boundary $\varepsilon_1 \times N$. Consider a double manifold constructed by glueing two copies of $M \setminus U_{\varepsilon_1}$ along the boundary. It is complete manifold without a boundary hence on this manifold $d_{min} = d_{max}$ ([LM], Theorem II.5.7). Thus

$$(1-\varphi)u \in D(d_{min}).$$

To prove that $\varphi u \in D(d_{min})$ we apply Lemma 3.7 from [B], which is true in particular for operators defined on M with such metric.

2) Follows directly from Lemma 3.2 because $d_{min} = d_{max}$. □

4 Asymptotic expansions for self-adjoint operators on a Riemannian manifold

The technique reviewed below can be applied to a compact set in a smooth Riemannian manifold. We use this methods to find the heat trace asymptotic on a regular part of a manifold with singularities.

4.1 The heat kernel expansion

Let (M, g) be a Riemannian manifold of dimension m. Consider operator $L := \frac{\partial}{\partial t} + \Delta_0$ where Δ_0 is Laplace-Beltrami operator on M.

Lemma 4.1. *There is a unique function on* $M \times M \times \mathbb{R}_+$ *which we denote* $e^{-t\Delta_0}(x, y)$ *such that*

- *it is* C^0 *in* x*,* C^2 *in* y *and* C^1 *in* t*;*
- *it satisfies the equation* $Le^{-t\Delta_0}(x, y) = 0$*, where Laplace operator is applied to the second variable;*
- $lim_{t \to 0+} e^{-t\Delta_0}(x, \cdot, t) = \delta_x$ *for any* $x \in M$*.*

For the proof of the existence and uniqueness see [BGM] Chapter III, E.II,E.III. $e^{-t\Delta_0}(x,y)$ is called the heat kernel of Laplace-Beltrami operator or simply the *heat kernel*.

Denote the distance between $x, y \in M$ by $d^2(x,y) = \sum_{i,j=1}^m g^{ij}x^i x^j =: r^2$ to find an expansion of $e^{-t\Delta_0}(x,y)$ we first introduce the following computational proposition.

Proposition 4.2. *Let* $h_t(x,y) = (4\pi t)^{-m/2} e^{\frac{-r^2}{4t}}$ *and* $g := \det(g_{ij})$*, then the following is true*

(a) $\frac{\partial}{\partial t} h_t(x,y) = (-\frac{m}{2t} + \frac{r^2}{4t^2}) h_t(x,y)$

(b) $dh_t(x,y) = -\frac{r}{2t} h_t(x,y) dr$

(c) $\Delta_0 h_t(x,y) = \left(\frac{m}{2t} + \frac{r}{4tg}\frac{\partial g}{\partial r} - \frac{r^2}{4t^2}\right) h_t(x,y)$

Proof. Recall that Laplace-Beltrami operator is given by $\Delta_0 = d_0^\dagger d_0$.

(a) is a partial differentiation of a function.

(b) $d\left((4\pi t)^{-m/2} e^{-r^2/4t}\right) = (4\pi t)^{-m/2}(-2\frac{r}{4t}) e^{-r^2/4t} dr = -\frac{r}{2t} h_t(x,y) dr.$

(c)$\Delta_0 h_t(x,y) = d_0^\dagger d_0 h_t(x,y) = -d_0^\dagger\left(\frac{r}{2t} h_t(x,y) dr\right)$

Next we use that for any function f and for any 1-form α the following is true $d_0^\dagger(f\alpha) = f d_0^\dagger \alpha - \langle d_0 f, \alpha\rangle$.

$$\begin{aligned}
\Delta_0 h_t(x,y) &= -\frac{1}{2t} h_t(x,y) d_0^\dagger(rdr) + \left\langle d_0\left(\frac{1}{2t} h_t(x,y)\right), rdr\right\rangle = \\
&= -\frac{1}{2t} h_t(x,y) d_0^\dagger(rdr) + \left\langle -\frac{r}{4t^2} h_t(x,y) dr, rdr\right\rangle = \\
&= -\frac{1}{2t} h_t(x,y) d_0^\dagger(rdr) - \frac{r^2}{4t^2} h_t(x,y)
\end{aligned} \tag{4.1}$$

the first term gives

$$d_0^\dagger(rdr) = -\frac{1}{\sqrt{g}} \sum_{j=1}^m \frac{\partial}{\partial x^j}(x^j \sqrt{g}) = -m - \frac{r}{\sqrt{g}}\frac{\partial g}{\partial r}.$$

Finally, we get the result

$$\Delta_0 h_t(x,y) = -\frac{1}{2t} h_t(x,y)\left(-m - \frac{1}{2g}\frac{\partial g}{\partial r}\right) - \frac{r^2}{4t^2} h_t(x,y).$$

□

Definition 2. Let B be a Banach space and $f(t) : [0, \infty) \to B$. A formal series

$$f(t) \sim_{t\to 0+} \sum_{k=0}^{\infty} a_k(t),$$

where $a_k(t) : [0, \infty) \to B$, is called *asymptotic expansion* of $f(t)$ near $t = 0$ if for any $n > 0$ there is l_n such that for any $l \geq l_n$ there is a constant $C_{l,n} > 0$ such that

$$||f(t) - \sum_{k=0}^{l} a_k(t)|| \leq C_{l,n} t^n$$

for small t.

Proposition 4.3. *Let $K \subset M$ be any compact set and let $x, y \in K$ be in one coordinate chart. There is an asymptotic expansion of the heat kernel*

$$e^{-t\Delta_0}(x, y) \sim_{t\to 0+} (4\pi t)^{-m/2} e^{-\frac{d^2(x,y)}{4t}} (u_0(x, y) + tu_1(x, y) + t^2 u_2(x, y) \dots)$$

where we have a recursive equation for $u_j(x, y)$

$$\frac{\partial u_j}{\partial r} + \left(\frac{1}{4g} \frac{\partial g}{\partial r} + \frac{j}{r} \right) u_j + \frac{1}{r} \Delta_0 u_{j-1} = 0$$

Proof. To simplify the notations we denote the infinite sum by $A := \sum_{j=0}^{\infty} t^j u_j(x, y)$, so $e^{-t\Delta_0}(x, y) \sim_{t\to 0+} h_t(x, y)A$ and

$$\left(\frac{\partial}{\partial t} + \Delta_0 \right) (h_t(x, y)A) = 0.$$

Using the Proposition 4.2 we obtain an equivalent equation

$$\left(-\frac{m}{2t} + \frac{r^2}{4t^2} \right) h_t(x, y)A + h_t(x, y) \frac{\partial}{\partial t} A + \Delta_0(h_t(x, y)A) = 0 \tag{4.2}$$

hence

$$\left(-\frac{m}{2t} + \frac{r^2}{4t^2} \right) A + \frac{\partial}{\partial t} A + \left(\frac{m}{2t} + \frac{r}{4tg} \frac{\partial g}{\partial r} - \frac{r^2}{4t^2} \right) A + \Delta_0 A + \frac{r}{t} \frac{\partial}{\partial r} A = 0.$$

Substitute the sum $A = \sum_{j=0}^{\infty} u_j(x, y) t^j$ into the equation above

$$\begin{aligned} \frac{r}{4g} \frac{\partial g}{\partial r} \sum_{j=0}^{\infty} u_j(x, y) t^{j-1} + \sum_{j=1}^{\infty} j u_j(x, y) t^{j-1} + \sum_{j=0}^{\infty} \Delta_0 u_j(x, y) t^j \\ + r \frac{\partial}{\partial r} \sum_{j=0}^{\infty} t^{j-1} u_j(x, y) = 0 \end{aligned} \tag{4.3}$$

The coefficient of t^{j-1} is

$$\frac{\partial u_j}{\partial r} + \left(\frac{1}{4g}\frac{\partial g}{\partial r} + \frac{j}{r}\right) u_j + \frac{1}{r}\Delta_0 u_{j-1}$$

and it should be equal to 0 for any $0 \leq j \leq \infty$. □

We are interested in the trace of the heat operator $tre^{-t\Delta_0} = \int_M e^{-t\Delta_0}(x,x)vol_M$ where vol_M is the volume form on M we would like to obtain information about functions $u_j(x,y)$ for $x = y$. Before we state the main result here are some notations concerning the curvature tensor and its derivatives. For a Riemannian manifold (M,g) we denote by D_X covariant derivative along the vector field X given by Levi-Civita connection. Now let $x \in M$ and $(x_1,\dots,x_m)$ be local coordinates near x and denote corresponding coordinate frame by $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}$. Define total covariant derivative $D := \sum_{i=1}^m D_{\frac{\partial}{\partial x_i}}$ and let D^k be the covariant derivative applied k times. Also denote the Riemannian curvature tensor at the point x by R_x or $g(R_x(X,Y)Z,T)$ for some $X,Y,Z,T \in T_xM$. The following proposition is a consequence of the Proposition 4.3.

Proposition 4.4. *Let $K \subset M$ be any compact set and $x \in K$. There is an asymptotic expansion of the heat kernel on the diagonal such that*

$$||e^{-t\Delta_0}(x,x) - (4\pi t)^{-m/2}\sum_{i=0}^{j} t^i u_i(x,x)|| \leq C(K)t^{j+1}$$

where $C(K)$ is some constant which depends on the compact set K. Moreover $u_0(x,x) = 1$ and $u_1(x,x) = \frac{1}{6}k(x)$ scalar curvature of M at a point x and all $u_i(x,x)$ are polynomials on curvature tensor and its covariant derivatives, i.e. for $i \geq 0$

$$u_i(x,x) = P_i^m(R_x, D^1R_x, D^2R_x, \dots).$$

where P_i^m is an $O(T_xM)$ invariant polynomial.

Remark. The last part of the proposition means that for any element of orthogonal group of isometries of T_xM $\sigma \in O(T_xM)$ we have

$$P_i^m(\sigma(R_x), \sigma(D^1R_x), \sigma(D^2R_x), \dots) = P_i^m(R_x, D^1R_x, D^2R_x, \dots).$$

for more details see [BGM] Prop E.VI.4.

4.2 The resolvent kernel expansion

The heat operator is closely related to the resolvent operator as we see below and if we have an asymptotic expansion of the kernel of one operator we could compute the coefficients in the expansion of the kernel of the other operator.

We introduce the following paths on a complex plane, defined for $0 < r \leq R \leq \infty$ and $0 \leq a < \pi$:

$$L_{r,R,a} := \{ye^{ia} : r \leq y \leq R\},$$

$$C_{r,a_1,a_2} := \{re^{i\alpha} : a_1 \leq \alpha \leq a_2\}.$$

Since the function $f(\lambda) = e^{-t\lambda}$ is holomorphic in a closed disk bounded by a circle $C_{R,\pi}$ for any $R > 0$, we can use the Chauchy differentiaton formula $f^{(m-1)}(\lambda) = \frac{(m-1)!}{2\pi i} \int_\gamma f(\mu)(\mu-\lambda)^{-m} d\mu$, where γ is a closed path on a complex plane passed counter-clockwise consisting of four pieces: $C_{\varepsilon,a,2\pi-a}$, $C_{R,-a,a}$ and two line segments $L_{\varepsilon,R,a}$ and $L_{\varepsilon,R,-a}$ then

$$e^{-t\lambda} = -t^{1-m} \frac{(m-1)!}{2\pi i} \int_\gamma e^{-t\mu}(\lambda - \mu)^{-m} d\mu. \tag{4.4}$$

We derive the expansion for the kernel of the resolvent $(\Delta_0 + z^2)^{-m}$ for an appropriate m using the Cauchy differentiation formula

$$e^{-t\Delta_0}(x,x) = -t^{1-m} \frac{(m-1)!}{2\pi i} \int_\gamma e^{-t\mu}(\Delta_0 - \mu)^{-m}(x,x) d\mu. \tag{4.5}$$

On a two-dimensional Riemannian manifold by the Proposition 4.4 we find the asymptotic expansion of the heat kernel. Assume that the expansion of the kernel of the resolvent has the form

$$(\Delta_0 - \mu)^{-m}(x,x) \sim_{|\mu| \to \infty} \sum_{k \in \mathbb{Z}} A_k(x,x)(-\mu)^k,$$

where $A_k(x,x)$ are some functions. The idea is to find the representation of $A_k(x,x)$ in terms of $u_j(x,x)$.

In the formula (4.5) let $t \to 0$ then we have an expansion on the left hand side

$$(4\pi t)^{-1} \sum_{j=0}^{\infty} u_j(x,x) t^j = -t^{1-m} \frac{(m-1)!}{2\pi i} \int_\gamma e^{-t\mu} \sum_{k \in \mathbb{Z}} A_k(x,x)(-\mu)^k d\mu$$

hence

$$(4\pi t)^{-1}\sum_{j=0}^{\infty} u_j(x,x)t^j = -t^{1-m}\frac{(m-1)!}{2\pi i}\sum_{k\in\mathbb{Z}} A_k(x,x)\int_\gamma e^{-t\mu}(-\mu)^{-k}d\mu,$$

$$(4\pi t)^{-1}\sum_{j=0}^{\infty} u_j(x,x)t^j = -t^{1-m}\frac{(m-1)!}{2\pi i}\sum_{k\in\mathbb{Z}} A_k(x,x)\frac{-2\pi i}{\Gamma(k)}t^{k-1},$$

$$\frac{1}{4\pi(m-1)!}\sum_{j=0}^{\infty} u_j(x,x)t^{j+m-2} = \sum_{k\in\mathbb{Z}} A_k(x,x)\frac{1}{\Gamma(k)}t^{k-1}.$$

Now equating the coefficients of the same powers of t we find $A_k(x,x)$

$$A_k(x,x) = \frac{\Gamma(k)}{4\pi(m-1)!}u_{k-m+1}(x,x).$$

Therefore we obtain the expansion of the kernel of the resolvent on a diagonal for some $x \in K \subset M$

$$(\Delta_0-\mu)^{-m}(x,x) \sim_{|\mu|\to\infty} (4\pi)^{-1}\sum_{j=m-1}^{\infty}(-\mu)^{-j}u_{j+1-m}(x,x)\frac{\Gamma(j)}{(m-1)!} \quad (4.6)$$

where $u_0(x,x) = 1$.

Consider a Riemannian manifold (M,g) such that its total curvature tensor is zero. By the Proposition 4.4 all $u_i(x,x)$ for $i \geq 1$ vanish and therefore we get

$$e^{-t\Delta_0}(x,x) \sim_{t\to 0+} (4\pi t)^{-m/2}.$$

Apply the result to the tetrahedron

$$e^{-t\Delta_0}(x,x) \sim_{t\to 0+} \frac{1}{4\pi}t^{-1}.$$

By the formula (4.6)

$$(\Delta_0-\mu)^{-m}(x,x) \sim_{|\mu|\to\infty} (-\mu)^{-m+1}\frac{1}{4\pi(m-1)} \quad (4.7)$$

5 Singular part

5.1 Regular singular operator

Consider a metric cone $U = I \times N := (0;\varepsilon) \times S^1$ with $r \in (0,\varepsilon)$, $\theta \in [0,2\pi)$ and the following metric

$$g_{cone} = dr^2 + r^2 a^2 d\theta^2,$$

where a is some positive constant.

First we show that Laplace-Beltrami operator on U is a second order regular singular operator according to (RS1)-(RS4) in [BS2]. Define a linear map

$$\Psi : C_c^\infty((0,\varepsilon), C^\infty(N)) \to C_c^\infty(U)$$

which acts as multiplication $f \mapsto r^{1/2}a^{1/2}f$ it extends to unitary map between Hilbert spaces

$$\Psi : L^2((0,\varepsilon), L^2(N)) \to L^2(U). \tag{5.1}$$

Given the metic we can write the operator $\Delta := d^\dagger d : C_c^\infty(U) \to C_c^\infty(U)$ in terms of partial derivatives with respect to local coordinates, we have $d : C_c^\infty(U) \to \lambda_c^1(U)$ and $d^\dagger = - * d* : \lambda_c^1(U) \to C_c^\infty(U)$ where $*$ is Hodge-star operator on U.

Let $f(r,\theta) \in C_c^\infty(U)$ since $\Delta = - * d * d$ we have

$$\begin{aligned}
f(r,\theta) &\overset{d}{\mapsto} \partial_r f dr + \partial_\theta f d\theta \\
&\overset{*}{\mapsto} -\frac{\partial_\theta f}{ra} dr + \partial_r f ra d\theta \\
&\overset{d}{\mapsto} \frac{\partial_\theta^2 f}{ra} dr \wedge d\theta + \left(\partial_r^2 f ra + \partial_r f a\right) dr \wedge d\theta \\
&\overset{*}{\mapsto} \frac{\partial_\theta^2 f}{r^2 a^2} + \partial_r^2 f + \frac{\partial_r f}{r} \\
&\overset{-}{\mapsto} \left(-\partial_r^2 - r^{-1}\partial_r - r^{-2}a^{-2}\partial_\theta^2\right) f(r,\theta)
\end{aligned} \tag{5.2}$$

The operator Δ acts on $C_c^\infty((0,\varepsilon) \times N)$ in the Hilbert space $L^2((0,\varepsilon) \times N)$.

Denote $\psi(r) := ra$ than we obtain

$$\Delta = -\psi^{-1}(\partial_r \psi \partial_r) - \psi^{-2}\partial_\theta^2$$

Apply the unitary transform (5.1) and define the operator acting on $C_c^\infty((0,\varepsilon), C^\infty(N))$ in the Hilbert space $L^2((0,\varepsilon), L^2(N))$

$$T := \Psi^{-1}\Delta\Psi.$$

Note that Ψ acts as $f \mapsto \psi^{1/2}(r) f$ and compute

$$\begin{aligned} T &= \psi^{1/2}(r) \Delta \psi^{-1/2}(r) \\ &= -\psi^{-1/2} \partial_r \psi (-\frac{1}{2} \psi^{-3/2} \psi' + \psi^{-1/2} \partial_r) - \psi^{-2} \partial_\theta^2 \\ &= -\frac{1}{4} \psi^{-2} (\psi')^2 + \frac{1}{2} \psi^{-1} \psi'' + \frac{1}{2} \psi^{-1} \psi' \partial_r - \frac{1}{2} \psi^{-1} \psi' \partial_r - \partial_r^2 - \psi^{-2} \partial_\theta^2 \\ &= -\partial_r^2 + \psi^{-2} (-\partial_\theta^2 - \frac{1}{4} (\psi')^2 + \frac{1}{2} \psi \psi'') \end{aligned} \tag{5.3}$$

Finally

$$T = -\partial_r^2 + r^{-2} (-a^{-2} \partial_\theta^2 - \frac{1}{4}) =: -\partial_r^2 + r^{-2} A. \tag{5.4}$$

Note that in case of tetrahedron we have a particular constant $a = 4$.

T is a non negative densely defined symmetric operator by the Theorem 2.3 it admits Friedrichs extension. Let T^F be the Friedrichs extension of T, below we deal only with Friedrichs extension so to simplify the notation denote it by T as well

$$T : L^2((0, \varepsilon), L^2(N)) \to L^2((0, \varepsilon), L^2(N)).$$

Denote $H := L^2((0, \varepsilon), L^2(N))$. Let $\varphi_1, \varphi_2 \in C_c^\infty(M)$ then

$$\varphi_1 T^{-1} \varphi_2 \in S_q(H),$$

where $S_q(H)$ is Schatten class of operators from the Definition 1 and $q > 0$. It follows from the resolvent equation and Hölder inequality for Schatten norms that

$$\varphi_1 (T + z^2)^{-m} \varphi_2 \in S_1(H)$$

for $m > q$ with uniform trace norm estimate in

$$\{ z \in \mathbb{C} \mid |arg z| < \delta \} \qquad 0 < \delta < \pi/2.$$

Moreover it has been shown in [BS] (pp. 400-409) that for any function $\varphi \in C_c^\infty(\mathbb{R})$ the operator $\varphi (T + z^2)^{-m}$ is trace class for $m > n/2$, where $n = dim M = 2$.

Let φ be a function on a manifold M with support near the singularity such that it depends only on the first coordinate r then by (4.6) for $x = (r, \theta) \in M$ we have

$$\varphi(r) (T + z^2)^{-m} (x, x) \sim_{|z| \to \infty} (4\pi)^{-1} \sum_{j=m-1}^{\infty} z^{-2j} \varphi(r) u_{j+1-m}(x, x) \frac{\Gamma(j)}{(m-1)!}.$$

5.2 Trace of the resolvent

As before $T = -\partial_r^2 + r^{-2}A$, where A is a self-adjoint operator on $L^2(N)$.

The resolvent of T is given by

$$(T + z^2)^{-1} = \otimes_{\lambda_k \in \text{Spec } A}(-\partial_r^2 + r^{-2}\lambda_k + z^2)^{-1} \otimes \pi_k,$$

where π_k is the projection on the k-th eigenspace of A. In other words, given an element $u(r) \in L^2((0,\varepsilon), L^2(N))$ we may write it as a sum $u(r) = \sum_{\lambda_k \in \text{Spec} A} c_k u_k(r)$, where c_k are some constants, $u_k(r) \in L^2((0,\varepsilon), L^2(N))$ and for each $r \in (0,\varepsilon)$ $u_k(r)$ are eigenvectors of A which make an orthogonal basis. Since $Au_k(r) = \lambda_k u_k(r)$, the action of the operator A on the corresponding eigenspace is given by λ_k.

Lemma 5.1. *Let $Im\ z^2 \neq 0$ and $0 < r_1 \leq r_2 < \infty$ then the resolvent $(-\partial_r^2 + r^{-2}\lambda_k + z^2)^{-1}$ is an integral operator with kernel given by*

$$(-\partial_r^2 + r^{-2}\lambda_k + z^2)^{-1}(r_1, r_2) = (r_1 r_2)^{1/2} I_\nu(r_1 z) K_\nu(r_2 z),$$

where $I_\nu(r_1 z)$ and $K_\nu(r_2 z)$ are modified Bessel functions, two linearly independent solutions of the modified Bessel equation.

Proof. Let $v_1(r,z)$ and $v_2(r,z)$ be solutions of

$$(-\partial_r^2 + r^{-2}\lambda_k + z^2)u(r) = 0 \tag{5.5}$$

then by the Theorem XIII.3.16 [DS] the resolvent is an integral operator and the kernel of the resolvent is

$$(-\partial_r^2 + r^{-2}\lambda_k + z^2)^{-1}(r_1, r_2) = (v_1' v_2 - v_1 v_2')^{-1}(r_1, z) v_1(r_1, z) v_2(r_2, z) \tag{5.6}$$

for $0 < r_1 < r_2 < \infty$.

We now find v_1 and v_2 by solving the equation (5.5). Put

$$u(r) =: r^{1/2} w(r)$$

then

$$\partial_r u(r) = \frac{1}{2} r^{-1/2} w(r) + r^{1/2} \partial_r w(r)$$

and

$$\partial_r^2 u(r) = -\frac{1}{4} r^{-3/2} w(r) + r^{-1/2} \partial_r w(r) + r^{1/2} \partial_r^2 w(r)$$

and (5.5) deformes to give the modified Bessel equation

$$\left(r^2 \partial_r^2 + r\partial_r - (r^2 z^2 + \lambda_k + \frac{1}{4}) \right) w(r) = 0.$$

Define $\nu := \sqrt{\lambda_k + \frac{1}{4}}$ then the general solution is generated by the modified Bessel functions

$$w(r) = C_1 I_\nu(rz) + C_2 K_\nu(rz)$$

where $C_1, C_2 \in \mathbb{R}$. Hence

$$u(r) = C_1 r^{1/2} I_\nu(rz) + C_2 r^{1/2} K_\nu(rz).$$

Note that modified Bessel function of the first kind $I_\nu(r)$ grows exponentially as $r \to \infty$ and $\nu > 0$ but tends to zero as $r \to 0$ on the other hand modified Bessel function of the second kind $K_\nu(r)$ tends to zero as $r \to \infty$ and grows as $r \to 0$. Using the boundary condition at $r = 0$ and the boundary condition at infinity $\lim_{r\to\infty} u(r) = 0$ we derive two linearly independent solutions of (5.5)

$$v_1(r) = r^{1/2} I_\nu(rz)$$

and

$$v_2(r) = r^{1/2} K_\nu(rz).$$

To find the kernel of the resolvent we use the formula (5.6). Since $I'_\nu(x)K_\nu(x) - I_\nu(x)K'_\nu(x) = x^{-1}$ then

$$\begin{aligned}
&(v_1' v_2 - v_1 v_2')^{-1}(r_1, z) \\
=&\left(\frac{1}{2} r_1^{-1/2} I'_\nu(r_1 z) + z r_1^{1/2} I'_\nu(r_1 z)\right) r^{1/2} K_\nu(r_1 z) \\
&- r_1^{1/2} I_\nu(r_1 z)\left(\frac{1}{2} r_1^{-1/2} K'_\nu(r_1 z) + z r_1^{1/2} K'_\nu(r_1 z)\right) \\
=&r_1 z\big(I'_\nu(r_1 z) K_\nu(r_1 z) - I_\nu(r_1 z) K'_\nu(r_1 z)\big) \\
=&1
\end{aligned} \tag{5.7}$$

By (5.6)

$$\begin{aligned}
&(-\partial_r^2 + r^{-2}\lambda_k + z^2)^{-1}(r_1, r_2) \\
=&(r_1 r_2)^{1/2} I_\nu(r_1 z) K_\nu(r_2 z)
\end{aligned} \tag{5.8}$$

□

From the Lemma 5.1 follows that the operator $(T + z^2)^{-1}$ has the following kernel on the diagonal $r_1 = r_2$

$$r \sum_{\lambda_k \in \mathrm{Spec} A} I_\nu(rz) K_\nu(rz).$$

Now we would like to obtain the trace of the resolvent. Following the discussion in the end of the Section 5.1 we recall that $(T+z^2)^{-1}$ is not trace class instead we consider the operator $(T+z^2)^{-m}, m>1$. To construct the trace of operator $(T+z^2)^{-m}$ first note that

$$(T+z^2)^{-m} = \frac{1}{(m-1)!}\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{m-1}(T+z^2)^{-1}.$$

Lemma 5.2. *([BS2] Lemma 4.3) Let $\varepsilon_0 < \varepsilon$ and $\varepsilon_1 > \varepsilon_0$ consider a function $\varphi(r) \in C_0^\infty(-1,\varepsilon_0)$ such that $\varphi(r) \equiv 1$ near $r=0$ and $\psi \in C_0^\infty(-1,\varepsilon)$ such that $\psi \equiv 1$ for $0 \le r \le \varepsilon_1$. Then we have for $m>1$*

$$\begin{aligned}
&tr\ (\psi(T+z^2)^{-m}\varphi) \\
&= \int_0^\infty \varphi(r)\frac{r^{2m-1}}{(m-1)!}\left(-\frac{1}{2rz}\frac{\partial}{\partial(rz)}\right)^{m-1}\sum_{\lambda_k\in SpecA} I_\nu(rz)K_\nu(rz)dr \\
&=: \int_0^\infty \varphi(r)\sum_\nu \sigma_\nu(r,rz)dr,
\end{aligned}$$

where $\nu = \sqrt{\lambda_k + \frac{1}{4}}$ and $\lambda_k \in SpecA$.

Note that if $\nu \to \infty$ then we can estimate the result (See [BS2] Lemma 3.2 p.385) therefore we consider a finite case when ν is a fixed number. Let $|z| \to \infty$ then there the following expansions ([AS] p.377-378)

$$I_\nu(rz) \sim \frac{1}{\sqrt{2\pi}}e^{rz}(rz)^{-1/2}\left(1-\frac{4\nu^2-1}{8rz}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8rz)^2}-\dots\right) \tag{5.9}$$

and

$$K_\nu(rz) \sim \sqrt{\frac{\pi}{2}}e^{-rz}(rz)^{-1/2}\left(1+\frac{4\nu^2-1}{8rz}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8rz)^2}-\dots\right). \tag{5.10}$$

Here the general coefficient of $(rz)^{-j}$ in brackets is 1 for $j=0$ and for $j>0$

$$\alpha_j(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\dots(4\nu^2-(2j-1)^2)}{j!8^j}$$

Finally we have

$$\sigma_\nu(r,\zeta) = \frac{r^{2m-1}}{(m-1)!}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1}I_\nu(\zeta)K_\nu(\zeta).$$

Now return to the Friedrichs extension of Laplace-Beltrami operator on the tetrahedron M, i.e. tetrahedron with singular points removed. Let $(r,\theta) = x \in K \subset M$ be a point at the compact subset of the tetrahedron and $\varphi(r) \in C_c^\infty((-\varepsilon,\varepsilon))$ then from the asymptotics of the resolvent of Laplace-Beltrami operator on the tetrahedron (4.7) we see

$$\varphi(r)(\Delta_0+z^2)^{-m}(x,x) \sim_{|z|\to\infty} \varphi(r) z^{-2m+2}\frac{1}{4\pi(m-1)}$$

hence

$$\begin{aligned} tr\varphi(r)(\Delta_0+z^2)^{-m} &= \int_M \varphi(r)(\Delta_0+z^2)^{-m}(x,x)vol_M \\ &= \int_0^\infty\int_0^{2\pi} \varphi(r)(\Delta_0+z^2)^{-m}(r,\theta,r,\theta)rad\theta dr \\ &\sim_{|z|\to\infty} \int_0^\infty\int_0^{2\pi} \varphi(r)z^{-2m+2}ra\frac{1}{4\pi(m-1)}d\theta dr \\ &= \int_0^\infty \varphi(r)z^{-2m+2}ra\frac{1}{2(m-1)}dr \end{aligned}$$

here we used $vol_M = radr\wedge d\theta$. Therefore

$$\sigma_\nu(r,\zeta) \sim_{\zeta\to\infty} \sigma_{\nu,2m-2}(r)\zeta^{-2m+2} = \frac{a}{2(m-1)}r^{2m-1}\zeta^{-2m+2}.$$

We can apply the regular asymptotics lemma $(Ia)-(Ic)$ from [?] to have the expansion of the integral

$$\begin{aligned} \int_0^\infty \varphi(r)\sigma_\nu(r,rz)dr \sim & \sum_{l=0}^\infty z^{-l-1}\frac{1}{l!}\int_0^\infty \zeta^l\partial_r^l\big(\sigma_\nu(r,\zeta)\varphi(r)\big)|_{r=0}d\zeta \\ & + z^{-2m+2}\frac{a}{2(m-1)}\int_0^\infty r\varphi(r)dr \\ & + \sum_{l=0}^\infty z^{-l-1}\frac{1}{l!}\log z\,\partial_r^l\big(\sigma_{\nu,l+1}(r)\varphi(r)\big)|_{r=0} \end{aligned}$$

Since

$$\begin{aligned} &\sum_{l=0}^\infty \partial_r^l\big(\sigma_\nu(r,\zeta)\varphi(r)\big)|_{r=0} \\ =&\frac{1}{(m-1)!}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu(\zeta)K_\nu(\zeta)\sum_{l=0}^\infty \partial_r^l\big(r^{2m-1}\varphi(r)\big)_{r=0} \\ =&\frac{(2m-1)!\varphi(0)}{(m-1)!}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu(\zeta)K_\nu(\zeta) \end{aligned}$$

and

$$\sum_{l=0}^{\infty} \partial_r^l \big(\sigma_{\nu,l+1}(r)\varphi(r)\big)|_{r=0} = \partial_r^{2m-3}(\sigma_{\nu,2m-2}(r)\varphi(r))|_{r=0} = 0$$

we find

$$\begin{aligned}
&\int_0^\infty \varphi(r)\sigma_\nu(r,rz)dr \sim \\
&+ z^{-2m}\frac{\varphi(0)}{(m-1)!}\int_0^\infty \zeta^{2m-1}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu(\zeta)K_\nu(\zeta)d\zeta \\
&+ z^{-2m+2}\frac{a}{2(m-1)}\int_0^\infty r\varphi(r)dr
\end{aligned}$$

hence

$$\begin{aligned}
&\mathrm{tr}(\varphi(\Delta_0+z^2)^{-m}) \sim \\
&+ z^{-2m}\frac{\varphi(0)}{(m-1)!}\sum_{\lambda_k\in \mathrm{Spec}A}\int_0^\infty \zeta^{2m-1}\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu(\zeta)K_\nu(\zeta)d\zeta \\
&+ z^{-2m+2}\frac{a}{2(m-1)}\int_0^\infty r\varphi(r)dr
\end{aligned}$$

Let's analyse the first summand. Denote $p := 2m-1$ then by the Mellin transform (see p.123 [O]) we obtain

$$\begin{aligned}
&\int_0^\infty \zeta^p\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{m-1} I_\nu(\zeta)K_\nu(\zeta)d\zeta \\
&= \frac{1}{4\sqrt{\pi}}\frac{\Gamma(\nu-m+\frac{p+3}{2})\Gamma(m-1-\frac{p}{2})\Gamma(\frac{p+1}{2})}{\Gamma(m+1+\nu-\frac{p+3}{2})}
\end{aligned}$$

substitute $p = l+2m-2$ where $l=1$

$$\frac{1}{4\sqrt{\pi}}\Gamma(-\frac{l}{2})\Gamma(m+\frac{l}{2}-\frac{1}{2})\frac{\Gamma(\nu+\frac{l}{2}+\frac{1}{2})}{\Gamma(\nu-\frac{l}{2}+\frac{1}{2})}$$

The ratio of two Gamma functions is given in the proposition below. Let B_j be Bernoulli numbers

$$B_0 = 1, B_j = -\sum_{i=0}^{j-1} C_j^i\frac{B_i}{j-i+1}.$$

Proposition 5.3.

$$\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \sim_{\nu\to\infty} \nu^{1-2s}\left(1 + s\sum_{j\geq 1}(-1)^{j-1}j^{-1}B_{j+1}\nu^{-2j}\right) + O(s^2)$$

Proof. According to [WW] Chapter XII p.251 (note that in the book the old notation of Bernoulli numbers is used, namely $B_1 = 1/6$ and so on but here we use the modern notation, i.e. $B_1 = -1/2$, $B_2 = 1/6$ that is why the formula is slightly different)

$$\log\Gamma(z) \sim_{z\to\infty} (z - \frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \sum_{j\geq 1}\frac{(-1)^{j-1}}{2j(2j-1)}\frac{B_{j+1}}{z^{2j-1}}$$

hence

$$\begin{aligned}\log\Gamma(\nu + s) \sim_{\nu\to\infty}&(\nu + s - \frac{1}{2})\log(\nu + s) - (\nu + s) + \frac{1}{2}\log(2\pi)\\ &+ \sum_{j\geq 1}\frac{(-1)^{j-1}}{2j(2j-1)}\frac{B_{j+1}}{(\nu + s)^{2j-1}}\\ =&(\nu + s - \frac{1}{2})(\log\nu + \log(1 + s/\nu)) - (\nu + s) + \frac{1}{2}\log(2\pi)\\ &+ \sum_{j\geq 1}\frac{(-1)^{j-1}}{2j(2j-1)}\frac{B_{j+1}}{(\nu + s)^{2j-1}}\end{aligned}$$

for the last equality use $\log(\nu + s) = \log\nu + \log(1 + s/\nu)$. Analogously use $\log(\nu - s) = \log\nu + \log(1 - s/\nu)$ to obtain

$$\begin{aligned}\log\Gamma(\nu - s) \sim&(\nu - s - \frac{1}{2})(\log\nu + \log(1 - s/\nu)) - (\nu - s) + \frac{1}{2}\log(2\pi)\\ &+ \sum_{j\geq 1}\frac{(-1)^{j-1}}{2j(2j-1)}\frac{B_{j+1}}{(\nu - s)^{2j-1}}\end{aligned}$$

Furthermore

$$
\begin{aligned}
&\log \frac{\Gamma(\nu-s)}{\Gamma(\nu+s)} = \log\Gamma(\nu-s) - \log\Gamma(\nu+s) \sim \\
&-2s\log\nu + (\nu - s - \frac{1}{2})\log(1-\frac{s}{\nu}) - (\nu+s-\frac{1}{2})\log(1+\frac{s}{\nu}) + 2s \\
&+\sum_{j\geq 1} \frac{(-1)^{j-1}}{2j(2j-1)} B_{j+1}\left(\frac{1}{(\nu-s)^{2j-1}} - \frac{1}{(\nu+s)^{2j-1}}\right) \\
\sim &-2s\log\nu + (\nu - s - \frac{1}{2})(-\frac{s}{\nu}) - (\nu+s-\frac{1}{2})(\frac{s}{\nu}) + 2s \\
&+\sum_{j\geq 1} \frac{(-1)^{j-1}}{2j(2j-1)} B_{j+1}\left(\frac{(\nu+s)^{2j-1} - (\nu-s)^{2j-1}}{(\nu^2-s^2)^{2j-1}}\right) + O(s^2) \\
\sim &-2s\log\nu + \frac{s}{\nu} + s\sum_{j\geq 1}(-1)^{j-1}j^{-1}B_{j+1}\nu^{-2j} + O(s^2).
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)} &= (\nu-s)\frac{\Gamma(\nu-s)}{\Gamma(\nu+s)} \\
&\sim (\nu-s)\nu^{-2s}\left(1 + \frac{s}{\nu} + s\sum_{j\geq 1}(-1)^{j-1}j^{-1}B_{j+1}\nu^{-2j}\right) + O(s^2) \\
&\sim \nu^{1-2s}\left(1 + s\sum_{j\geq 1}(-1)^{j-1}j^{-1}B_{j+1}\nu^{-2j}\right) + O(s^2)
\end{aligned}
$$

□

By the Proposition 5.3 we may write

$$
\begin{aligned}
b_{2m} &:= \sum_{\lambda_k\in\mathrm{Spec}A} \frac{\Gamma(-\frac{l}{2})\Gamma(m+\frac{l}{2}-\frac{1}{2})}{4\sqrt{\pi}(m-1)!} \frac{\Gamma(\sqrt{\lambda_k+\frac{1}{4}}+\frac{l}{2}+\frac{1}{2})}{\Gamma(\sqrt{\lambda_k+\frac{1}{4}}-\frac{l}{2}+\frac{1}{2})} \\
&= \frac{\Gamma(-\frac{l}{2})\Gamma(m+\frac{l}{2}-\frac{1}{2})}{4\sqrt{\pi}\Gamma(m)} \times \\
&\times \sum_{\lambda_k\in\mathrm{Spec}A}\left(\left(\sqrt{\lambda_k+\frac{1}{4}}\right)^{l} + \sum_{j\geq 1}(-1)^j j^{-1}B_{j+1}(\frac{l}{2}-\frac{1}{2})\left(\sqrt{\lambda_k+\frac{1}{4}}\right)^{-2j+l}\right).
\end{aligned}
$$

To find the regular analytic continuation at $l=1$ of the function above we set some notations. Denote $w := -l$ and let f be a meromorphic function with Laurent series

$$f(z) = \sum \mathrm{Res}_n f(z_0)(z - z_0)^{-n},$$

where $\mathrm{Res}_n f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{(z-z_0)^{n+1}}$ with γ a counterclockwise path of integration enclosing z_0 and lying in an annulus in which $f(z)$ is holomorphic. By these notations Res_1 is the residue of the function and Res_0 is the regular analytic continuation. Let f be analytic and g have a simple pole at z_0, then

$$\mathrm{Res}_0(fg)(z_0) = f(z_0)\mathrm{Res}_0 g(z_0) + f'(z_0)\mathrm{Res}_1 g(z_0).$$

Set

$$f(w) := \frac{\Gamma(\frac{w}{2})\Gamma(m - \frac{w}{2} - \frac{1}{2})}{4\sqrt{\pi}\Gamma(m)}$$

and

$$\zeta_A(w) := \sum_{\lambda_k \in \mathrm{Spec}A} \left(\sqrt{\lambda_k + \frac{1}{4}} \right)^{-w}, \qquad g(w) := \sum_{j \geq 1} (-1)^j j^{-1} B_{j+1} \zeta_A(2j + w)$$

since $f(1) = -\frac{1}{2}$ and $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $f'(1) = -\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} - \frac{\Gamma'(m)}{4\Gamma(m)}$ we have

$$\begin{aligned} b_{2m} &= \mathrm{Res}_0 \left(f(w)\zeta_A(w) + f(w)(-\frac{w}{2} - \frac{1}{2})g(w) \right) \\ &= -\frac{1}{2}\mathrm{Res}_0\zeta_A(-1) - \left(\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(m)}{4\Gamma(m)} \right) \mathrm{Res}_1\zeta_A(-1) - \frac{1}{4}\mathrm{Res}_1 g(-1) \end{aligned}$$

Finally

$$\begin{aligned} b_{2m} = &-\frac{1}{2}\mathrm{Res}_0\zeta_A(-1) - \left(\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(m)}{4\Gamma(m)} \right) \mathrm{Res}_1\zeta_A(-1) \\ &- \frac{1}{4} \sum_{j \geq 1} (-1)^j j^{-1} B_{j+1} \mathrm{Res}_1 \zeta_A(2j - 1). \end{aligned} \tag{5.11}$$

Below is the final formula for the expansion of the resolvent of the Laplace-Beltrami operator on the tetrahedron which we use later for the expansion of the trace of the heat operator

Proposition 5.4. *Let $\varphi \in C_0^\infty(-1, \infty)$ such that $\varphi(r) \equiv 1$ near $r = 0$ and b_{2m} as in (5.11) then*

$$tr(\varphi(\Delta_0 + z^2)^{-m}) \sim z^{-2m} b_{2m} + z^{-2m+2} \frac{a}{2(m-1)} \int_0^\infty r\varphi(r)dr \tag{5.12}$$

5.3 Trace of the heat operator

Recall that the integral kernel of the heat operator and the integral kernel of the resolvent operator are closely related by the Cauchy differentiation formula (4.5) from which be obtain

$$tr\varphi(r)e^{-t\Delta_0} = -t^{1-m}\frac{(m-1)!}{2\pi i}\int_\gamma e^{-t\mu}\mathrm{tr}(\varphi(r)(\Delta_0-\mu)^{-m})d\mu, \tag{5.13}$$

where γ is a closed contour specified in (4.5).

Let $-\mu =: z^2$. In the previous section we have found the expansion of $\mathrm{tr}(\varphi(r)(\Delta_0+z^2)^{-m})$ see Proposition 5.4. To make the next discussion clear we compute the following integrals on the complex plane

$$\begin{aligned}\int_\gamma e^{-t\mu}(-\mu)^{-n}d\mu &= 2\pi i\mathrm{Res}_{\mu=0}(e^{-t\mu}(-\mu)^{-n})\\ &= \frac{2\pi i}{(n-1)!}\lim_{\mu\to 0}\left(\frac{d}{d\mu}\right)^{n-1}(-1)^n e^{-t\mu} = -\frac{2\pi i}{\Gamma(n)}t^{n-1}\end{aligned} \tag{5.14}$$

$$\begin{aligned}\int_\gamma e^{-t\mu}(-\mu)^{-n}\log(-\mu)d\mu &= -\frac{d}{dn}\int_\gamma e^{-t\mu}(-\mu)^{-n}d\mu = \frac{d}{dn}(\frac{2\pi i}{\Gamma(n)}t^{n-1})\\ &= 2\pi i\frac{t^{n-1}\log t\Gamma(n) - t^{n-1}\Gamma(n)'}{\Gamma(n)^2}\\ &= \frac{2\pi i}{\Gamma(n)}t^{n-1}\log t - \frac{2\pi i}{\Gamma(n)}\frac{\Gamma'(n)}{\Gamma(n)}t^{n-1}\end{aligned} \tag{5.15}$$

Note that $\Gamma(\frac{1}{2}+n) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

Following these observations we can derive the asymptotic expansion of the heat trace from (5.13) and (5.12). The coefficient of z^{-2m-2} gives a contribution to the coefficient of t^{-1} and the coefficient of z^{-2m} to t^0.

Now combine the formulas (5.12), (5.13) and (5.14) to find the coefficients in our case explicitly

$$\begin{aligned}tr\varphi(r)e^{-tT} \sim\ & t^{-1}\frac{a}{2}\int_0^\infty r\varphi(r)dr\\ &+ t^0\Bigg(-\frac{1}{2}\mathrm{Res}_0\zeta_A(-1) - \left(\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(m)}{4\Gamma(m)}\right)\mathrm{Res}_1\zeta_A(-1)\\ &- \frac{1}{4}\sum_{j\geq 1}(-1)^j j^{-1}B_{j+1}\mathrm{Res}_1\zeta_A(2j-1)\Bigg)\end{aligned} \tag{5.16}$$

where $\zeta_A(w)=\sum_{\lambda_k\in \mathrm{Spec} A}\left(\sqrt{\lambda_k+\frac{1}{4}}\right)^{-w}$, B_j are Bernoulli numbers and notations Res_0, Res_1 correspond to the regular analytic continuation and the residue respectively.

Compute the constant term. Recall $\Delta_0=-\partial_r^2+r^{-2}A$ with $A=-a^{-2}\partial_\theta^2-\frac{1}{4}$ that gives $\mathrm{Spec}\ A=\{a^{-2}k^2-\frac{1}{4}\mid k\in\mathbb{Z}\setminus\{0\}\}$ hence

$$\zeta_A(w)=2\sum_{k\in\mathbb{N}}\left(\frac{k}{a}\right)^{-w}=2a^w\zeta(w),$$

where $\zeta(w)$ is Riemann zeta function.

By regular analytic continuation of Riemann zeta function we obtain

$$\mathrm{Res}_0\zeta_A(-1)=-2a^{-1}\frac{1}{12}=-\frac{1}{6a}$$

and

$$\mathrm{Res}_1\zeta_A(2j-1)=\mathrm{Res}\zeta(2j-1)=0 \text{ for } j\geq 2$$
$$\mathrm{Res}_1\zeta_A(1)=2a\mathrm{Res}\zeta(1)=2a$$

therefore the constant term in the expansion equals

$$\frac{1}{12a}-\frac{a}{12}.$$

In case of the tetrahedron $a=\frac{1}{2}$ hence the constant term is $\frac{1}{8}$ in the neighbourhood of each singular point. Note that this term in the expansion of $tr\varphi(r)e^{-t\Delta_0}$ depend only on the value of the function $\varphi(r)$ at $r=0$ and we assumed that $\varphi(r)=1$ near $r=0$.

For the partition of unity φ_i,ψ with $1\leq i\leq 4$ chosen in the introduction we have

$$tr\varphi_i e^{-t\Delta_0}\sim_{t\to 0+}t^{-1}\frac{1}{4}\int_0^\infty r\varphi_i(r)dr+\frac{1}{8}=t^{-1}\frac{1}{4\pi}\int_M\varphi_i(r)vol_M+\frac{1}{8}$$

here we used that $vol_M=\frac{1}{2}rdr\wedge d\theta$. Also

$$tr\psi e^{-t\Delta_0}\sim_{t\to 0+}t^{-1}\frac{1}{4\pi}\int_M\psi vol_M$$

therefore

$$\begin{aligned}tre^{-t\Delta_0}=&tr\psi e^{-t\Delta_0}+\sum_{i=1}^4 tr\varphi_i e^{-t\Delta_0}\sim_{t\to 0+}\\ \sim&t^{-1}\frac{1}{4\pi}\int_M(\sum_{i=1}^4\varphi_i(r)+\psi)vol_M+\frac{1}{2}\\ =&t^{-1}\frac{\sqrt{3}}{4\pi}+\frac{1}{2}\end{aligned}$$

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